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# Powers of derivations on semiprime rings

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## Abstract :

Chung and Luh [1] studied semiprime rings with nilpotent derivatives and established the result for ($n$-1)!-torsion free semiprime rings. Giambruno and Herstein [2] proved the same result without assuming that $R$ is ($n$-1)!-torsion free. Bresar [3] generalized the result of Chung and Luh. Herstein proved some related results in [4] and [5]. In this paper we prove that if $R$ is an ($n$-1)!-torsion free semiprime ring with a derivaiton $d$ such that $bd(x)^n a = 0$ for $a,b \in R$ and for all $x \in R$, then $bd(x)a = 0$ for all $x \in R$.


Key words : Prime ring, semiprime ring, derivative, 2-torsion free ring.

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## I. Introduction :

We know that an additive map $d$ from a ring R to R is called a derivation fon R if $d(xy) = d(x)y + xd(y)$ for all $x,y$ in R. A ring R is called prime if and only if $xay = 0$ for all $a$ in R implies $x = 0$ or $y = 0$ and semiprime if and only if $xax = 0$ for all a in R implies $x = 0$.

Throughout this paper $R$ denotes an ($n$-1)!-torsion free ring with a derivation $d$ such that $bd(x)^n a = 0$.

## II.Main Theorem :

To prove the main Theorem we need the following Lemmas.

**Lemma 1.1:** Let $R$ be a $m$!-torsion free ring. Suppose that $t_1, t_2 \ldots. t_m \in R$ satisfy $kt_1 + k^2 t_2 + \ldots\ldots\ldots\ldots\ldots k^m t_m = 0$ for $k = 1,2\ldots\ldots..m$. Then $t_i = 0$ for all $i$.

**Lemma1.2:** For all $x,y \in R$,

$$\sum_{k=0}^{n-1} d(x)^k d(y) d(x)^{n-k-1} a = 0. \qquad 1.1$$

Using these, we prove the following.

**Lemma 1.3:** For all $x,y \in R$, $d^2(x)ybd(x)^{n-1}a = 0$.

**Proof:** Replacing $y$ by $d(x)y$ in 1.1, we obtain

$$\sum_{k=0}^{n-1} d(x)^k d(d(x)y) d(x)^{n-k-1} a = 0,$$

$$\sum_{k=0}^{n-1} d(x)^k (d^2(x)y + d(x)d(y)) d(x)^{n-k-1} a = 0,$$

$$\sum_{k=0}^{n-1} d(x)^k d^2(x) y d(x)^{n-k-1} a +$$

$$d(x) \sum_{k=0}^{n-1} d(x)^k d(y) d(x)^{n-k-1} a = 0 \cdot$$

Using the relation 1.1, this reduces to

$$\sum_{k=0}^{n-1} d(x)^k d^2(x) y d(x)^{n-k-1} a = 0, \text{ for all } x,y \in R \quad 1.2$$

Replacing $y = ybd(x)^{n-1}$ in the relation 1.2, we get

$$\sum_{k=0}^{n-1} d(x)^k d^2(x) ybd(x)^{e(n-k)-1} a = 0.$$

Since $bd(x)^n a = 0$, we get $d(x)^{n-1} d^2(x) ybd(x)^{n-1} a = 0$.

We will prove this Lemma by showing that

$$d(x)^{r+1} d^2(x)\, ybd(x)^{n-1} a = 0,$$

where $r \geq 0$ is any integer, which implies

$$d(x)^r d^2(x) ybd(x)^{n-1} = 0.$$

Taking $y = ybd(x)^r$ in the relation 1.2, we obtain

$$\sum_{k=0}^{n-1} d(x)^k d^2(x) ybd(x)^{n-k-1+r} a = 0.$$

Since $bd(x)^n a = 0$, this relation reduces to

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$$d(x)^r d^2(x)ybd(x)^{n-1}a + \sum_{k=r+1}^{n-1} d(x)^k d^2(x)ybd(x)^{n-k-1+r} a = 0.$$

Hence if $u$ is an arbitrary element in $R$, then

$$(d(x)^r d^2(x)ybd(x)^{n-1}a)u(d(x)^r d^2(x)ybd(x)^{n-1}a) =$$
$$- \sum_{k=r+1}^{n-1} d(x)^k d^2(x)ybd(x)^{n-k-1+r} au\,(d(x)^r d^2(x)ybd(x)^{n-1}a)$$
$$= - \sum_{k=r+1}^{n-1} d(x)^k d^2(x)ybd(x)^{n-k-1+r} aud(x)^r d^2(x)ybd(x)^{n-1}a$$

= 0. by hypothesis.

By semiprimeness of $R$, this relation implies that

$$d(x)^r d^2(x)ybd(x)^{n-1}a = 0. \quad \square$$

**Lemma 1.4:** For all $x,y,z \in R$,

$$d^2(z)ybd(x)^{n-1}a = 0 \qquad 1.3$$

**Proof:** By Lemma 1.3 we have

$$d^2(x)ybd(x)^{n-1}a = 0.$$

Linearizing, we obtain

$$T(x,z) = d^2(x+z)ybd(x+z)^{n-1}a = 0.$$

That is, $(d^2(x)+d^2(z))yb\,(d(x) + d(z))^{n-1}a = 0$.

Let us take $(d(x) + d(z))^{n-1}$ as $\gamma_0 + \gamma_1 + \ldots\ldots\ldots\ldots + \gamma_{n-1}$ where $\gamma_j$ denotes the sum of these terms in which $d(x)$ appears as a factor in the product $j$ times.

Since $d^2(x)ybd(x)^{n-1}a = d^2(z)ybd(x)^{n-1}a = 0$, we have

$$T(x,z) = \sum_{k=0}^{n-2} d^2(x)yb\gamma_k a + \sum_{j=1}^{n-1} d^2(z)yb\gamma_j a = 0.$$

Thus if $t_k = d^2(x)yb\gamma_{k-1}a + d^2(z)yb\gamma_k a$, then we can write

$$T(x,z) = t_1 + \ldots\ldots\ldots\ldots + t_{n-1}.$$

Clearly $T(kx,z) = kt_1 + k^2t_2 + \ldots\ldots\ldots.. + k^{n-1}t_{n-1}$ for every integer $k$.

Since $T(kx,z) = 0$, for $k = 1\ldots\ldots.n-1$, we have $t_{n-1} = 0$ by Lemma 1.1.

Note that $\gamma_{n-1} = d(x)^{n-1}$.

Thus $0 = t_{n-1} = d^2(x)yb\,\gamma_{n-2}a + d^2(z)yb\,\gamma_{n-1}a$
$\qquad = d^2(x)yb\,\gamma_{n-2}a + d^2(z)ybd(x)^{n-1}a$.

Using this relation and Lemma 1.3, for every $u \in R$ we have

$(d^2(z)ybd(x)^{n-1}a)ud^2(z)ybd(x)^{n-1}a =$
$(-d^2(x)yb\gamma_{n-2}a)\,u\,(d^2(z)ybd(x)^{n-1}a) =$
$-d^2(x)(yb\gamma_{n-2}aud^2(z)y)bd(x)^{n-1}a = 0$.

Hence $d^2(z)ybd(x)^{n-1}a = 0$ by the semiprimeness of $R$. $\square$

**Lemma 1.4:** For all $x \in R$, $bd(x)^2a = 0$.

**Proof:** We replace $z$ by $x^2$ in the relation 1.3.

Then $\quad d^2(x^2)\,yb\,d(x)^{n-1}a = 0$.
This implies $\quad d(d^2(x^2))\,yb\,d(x)^{n-1}a = 0$.
So $\ d(d(x)x + xd(x))\,yb\,d(x)^{n-1}a = 0$,
$(d^2(x)x + xd(x)^2 + xd^2d(x))\,yb\,d(x)^{n-1}a = 0$,
$[d^2(x)x + 2(d(x))^2 + xd^2(x)]\,yb\,d(x)^{n-1}a = 0$.
By Lemma 1.3, this relation reduces to

$$2d(x)^2\,yb\,d(x)^{n-1}a = 0.$$

Let us assume that $n \geq 3$.
Then $R$ is 2-torsion free by assumption.
So $d(x)^2\,yb\,d(x)^{n-1}a = 0$.
Since $y$ is arbitrary, we also have

$$d(x)^{n-1}aybd(x)^{n-1}a = 0.$$

Hence $bd(x)^{n-1}aybd(x)^{n-1}a = 0$.
By semiprimeness of $R$, we obtain

$$bd(x)^{n-1}a = 0.$$

Since $n$ is any integer larger than 2 we have by induction $bd(x)^2a = 0$. $\square$

**Theorem 1.1:** If $R$ is a semiprime ring with a derivation $d$ such that $bd(x)^na = 0$ for all $a,b,x \in R$ and $n$ is a positive integer, then $bd(x)a = 0$ for all $a,b,x \in R$. Moreover, if $R$ is prime, then either $a = 0$ or $b = 0$ or $d = 0$.

**Proof:** Let us assume that $bd(x)^na = 0$ for all $x,a,b \in R$. By lemma 1.4, we may assume that $n = 2$.

Hence by the relation 1.3, we have $d^2(z)ybd(x)a = 0$, for all $x,y,z \in R$.

Since $y$ is arbitrary, we have $bd^2(z)aybd^2(x)a = 0$.

In particular, $bd^2(x)aybd^2(x)a = 0$

and also $bd^2(z)d(x)aybd^2(z)d(x)a = 0$

which imply $bd^2(x)a = 0$, for all $x \in R$ and $\qquad 1.4$

$bd^2(z)d(x)a = 0$, for all $x,z \in R \qquad 1.5$

by the semiprimeness of $R$.

We linearize $bd^2(x)a = 0$. Then we get

$$bd(x+y)^2a = 0.$$

That is, $b[d(x) + d(y)]^2a = 0$ which implies
$bd(x)^2a + bd(y)^2a + bd(x)d(y)a + bd(y)d(x)a = 0$.
Using the equation 1.4, we obtain
$bd(x)d(y)a + bd(y)d(x)a = 0$, for all $x,y \in R$. $\qquad 1.6$

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By replacing $y$ by $ybd(x)$ in the equaion 1.6, we get

$$bd(x)d(ybd(x))a + bd(ybd(x))d(x)a = 0.$$

This implies

$bd(x)d(y)bd(x)a+bd(x)yd(b)d(x)a+bd(x)ybd^2(x)a+ bd(y)bd(x)^2a+byd(b)d(x)^2a+bybd^2(x)d(x)a = 0.$

Now, using the equations 1.4, 1.5 and $bd(x)^2a= 0$, this relation reduces to $bd(x)d(y)bd(x)a+ bd(x)yd(b)d(x)a + byd(b)d(x)^2a = 0.$

Replacing $b$ by $byd(b)$ in $bd(x)^2a = 0$, we get $byd(b)d(x)^2a = 0.$ 1.7

Hence $bd(x)[d(y)b + yd(b)]d(x)a = 0$ implies

$bd(x)d(yb)d(x)a=0$, for all $x,y\in R$. 1.8

Linearizing the equation 1.8, we obtain

$$bd(x+z)\ d(yb)\ d(x+z)a = 0,$$

$bd(x)d(yb)\ d(x)a + bd(z)d(yb)d(x)a + bd(x)d(yb)d(z)a + bd(z)d(yb)d(z)a = 0.$

Using the equation 1.8, we get,

$bd(x)d(yb)\ d(z)a + bd(z)d(yb)d(x)a = 0.$ 1.9

By taking $yb = ybd(z)$ in the equation 1.9, we get

$bd(x)d(ybd(z))d(z)a + bd(z)d(ybd(z))d(x)a = 0.$

This implies

$bd(x)d(y)bd(z)^2a+bd(x)yd(b)d(z)^2a+bd(x)yb\ d^2(z)d(x)a+bd(z)d(y)bd(z)d(x)a+bd(z)yd(b)d(z)d(x)a + bd(z)\ ybd^2(z)d(x)a = 0.$

Using the equation 1.5 and $bd(z)^2\ a= 0$, we obtain

$$bd(z)d(y)bd(z)d(x)a + bd(z)yd(b)d(z)d(x)a + bd(x)yd(b)d(z)^2a = 0.$$

Replacing y by d(x)y in the relation 1.7, we get

$$bd(x)yd(b)d(z)^2a = 0.$$

Therefore $bd(z)(d(yb + yd(b))d(z)d(x)a = 0.$

Hence $bd(z)d(yb)d(z)d(x)a = 0.$

Put $yb = ybd(x)u$ in this equation.

Then we have

$$bd(z)\ d(ybd(x)u)\ d(z)\ d(x)a = 0.$$

That is,

$bd(z)[d(y)bd(x)u+yd(b)d(x)u+ybd^2(x)d(u)+ \quad + ybd(x)d(u)]d(z)d(x)a = 0,$

$bd(z)d(y)bd(x)ud(z)d(x)a+bd(z)yd(b)d(x)ud(z)d(x)a +bd(z)ybd^2(x)d(u)d(z)d(x)a+ bd(z)ybd(x)d(u)d(z)d(x)a = 0.$ 1.10

By replacing $y$ by $d(u)z$ in the equation 1.8, we obtain

$$bd(x)d(d(u)zb)\ d(x)a = 0$$

$$bd(x)d^2(u)zbd(x)a + bd(x)d(u)d(zb)d(x)a = 0.$$

Using the equation 1.3, it reduces to

$$bd(x)d(u)d(zb)d(x)a = 0.$$

The equation 1.10 reduces to

$bd(z)d(yb)d(x)ud(z)d(x)a = 0,$ 1.11

for all $x,y,z,u \in Z$.

By replacing $b$ by $bd(yb)$ in the equation 1.6

$$bd(yb)d(x)d(yb)a + bd(y)^2d(x)a = 0.$$

By the relation 1.7, it follows that $bd(yb)^2d(x)a = 0$ for all $x,y \in R$.

On linearizing we get $bd(yb+z)^2\ d(x)a = 0,$

$bd(yb)^2d(x)a + bd(z)^2d(x)a + bd(yb)d(z)d(x)a + bd(z)\ d(yb)d(x)a = 0.$

Using the equation 1.5, it reduces to

$$bd(yb)d(z)d(x)a + bd(z)d(yb)d(x)a = 0.$$

Since the element $u$ the equation 1.11 is arbitrary, we also have

$$bd(z)d(yb)d(x)au\ bd(y)d(z)d(x)a = 0.$$

Combining these two relations,

$bd(z)d(yb)d(x)aubd(z)d(yb)d(x)a = 0$, for all $x,y,z \in R$.

Since $R$ is semiprime this relation implies

$bd(z)d(yb)d(x)a = 0$, for all $x,y,z \in R$. 1.12

By replacing $d(z)$ by $xd(z)$, we get

$bxd(z)d(yb)d(x)a=0.$ 1.13

By substituting $xz$ for $z$ in the equation 1.12, we obtain

$$b\ d(xz)\ d(yb)\ d(x)a = 0.$$

This implies

$$bd(x)zd(yb)d(x)a + b\ xd(z)d(yb)d(z)a = 0.$$

Hence $bd(x)zd(yb)d(x)a = 0$, for all $x,y,z \in R$ by using the equation 1.12

which yields

$bd(yb)d(x)a = 0$, since $R$ is semiprime.

Now, by replacing $yb$ by $xyb$, we get

$$bd(xyb)\ d(x)a = 0,$$

$bd(x)ybd(x)a + bxd(yb)\ d(x)a = 0.$ 1.14

By replacing $d(yb)$ by $xd(yb)$ in the equation 1.14, we get $bxd(yb)d(x)a = 0$, hence the above equation reduces to

$$bd(x)ybd(x)a = 0.$$

Since $y$ is arbitrary, we have

$$bd(x)aybd(x)a = 0.$$

Hence $bd(x)a = 0.$

If $R$ is prime then either $bd(x) = 0$ or $a = 0$ for all $x \in R$. Again by primeness of $R$ we get either $a= 0$ or $b = 0$ or $d(x) = 0.$

The proof of Theorem 1.1 is thus completed. ◻

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